NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2811

ON THE CALCULATION OF FLOW ABOUT OBJECTS TRAVELING AT HIGH SUPERSONIC SPEEDS

By A. J. Eggers, Jr.

Ames Aeronautical Laboratory
Moffett Field, Calif.

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SUMMARY

A procedure for calculating three-dimensional steady and nonsteady supersonic flows with the method of characteristics is developed and discussed. The flow is assumed to be adiabatic and inviscid, although it may be rotational and the gas may exhibit both thermal and caloric imperfections. The latter features of generality are retained in the analysis since it is known that the phenomena associated therewith may significantly influence flows at high supersonic airspeeds. A further study of the compatibility equations holding along characteristic lines reveals that at Mach numbers sufficiently large compared to 1, flow in the osculating planes of streamlines may, in regions free of shock waves, often be of the generalized Prandtl-Meyer type. Surface streamlines may, under such circumstances, be approximated by geodesics. These results hold for nonsteady as well as steady flow, provided only slender shapes are considered, and provided the induced curvature of the flow associated with the nonsteady motions does not exceed in order of magnitude the total curvature of the flow. Steady two-dimensional-flow equations may thus be applicable to a wider class of flows, and hence shapes, at high supersonic speeds than was heretofore thought.

INTRODUCTION

The calculation of flows about objects, primarily missiles, traveling at high supersonic speeds is now generally accepted as a matter of more than academic interest. The difficulty of these calculations stems in large part from the fact that at such high speeds disturbance velocities are not necessarily small compared to the velocity of sound, nor are entropy gradients necessarily negligible in the disturbed flow field about a body, even though it may be of normal slenderness. Thus, for example, the relatively simple linear theory, which has proven so valuable in studying flows at low supersonic speeds, loses much of its utility in the study of high-supersonic-speed flows. In the quest for methods especially suited to calculating high-supersonic-speed flows, notable progress has been made in the development of similarity laws relating the flows about slender three-dimensional shapes in both steady (see references 1, 2, and 3) and nonsteady motion (see references 4 and 5). Steady two-dimensional flows have received perhaps the greatest attention from the standpoint of calculating specific flow fields, and it would seem that with tools ranging from the characteristics method (see, e.g., references 6 and 7) to the generalized shock-expansion method (reference 7) the problem is reasonably well in hand, at least insofar as inviscid, continuum flow is concerned. A more or less analogous situation exists with regard to the nonlifting body of revolution (see, e.g., references 6, 8, 9, and 10) although it seems that only in the case of the cone has a method (reference 10) of simplicity comparable to that of the linear theory been developed for calculating the whole flow field.

When one departs from these relatively simple flows, the number of tools for carrying out practical calculations decreases sharply. Thus, for example, in the category of inclined bodies of revolution, it appears that only bodies at small angles of attack have been handled adequately, usually by either the method of characteristics or some other step-by-step calculative procedure (see, e.g., references 6, and 11 through 14). In the case of steady flow about general three-dimensional shapes, aside from Newtonian flow concepts, which are strictly applicable at Mach numbers exceeding all limits, only the characteristics method has apparently thus far received serious attention (references 15, 16, 17, and 18). Sauer's treatment of this method (reference 18) is especially neat, entailing only the assumption of ideal (i.e., thermally and calorically perfect) gas flow and yielding compatibility equations in a relatively simple form. Application of the method, although it would undoubtedly prove tedious and time consuming, is, as pointed out by Sauer, certainly feasible with the aid of present day high-speed computing machines. It is clear, of course, that the relatively exact solutions obtainable with the method of characteristics provide an invaluable check against the predictions of more approximate but perhaps simpler theories. Indeed, as demonstrated in reference 10, a study of the compatibility equations of the characteristics method can prove useful in determining simplified methods for calculating more complex flow fields.

With these points in mind, it is undertaken as the first principal objective of the present report to reconsider the characteristics method, particular attention being given to its application to high-supersonic-speed flows in which, as is often the case, air does not behave as an ideal gas. The assumption, then, of ideal gas flow is relaxed, and to preserve insofar as is possible the element of simplicity in the method, pressure and flow inclination angles are employed as dependent variables (see reference 7) rather than, for example, velocity components as were used by Sauer. Extension of the method to the study of nonsteady flows is also considered; however, the remaining principal objective of this paper is to show how results of the characteristics theory can be exploited to deduce a simplified procedure for calculating high-supersonic-speed flows of both the steady and nonsteady types.

ANALYSIS AND DISCUSSION

Compatibility relations describing the behavior of fluid properties along characteristic lines in supersonic flow may, of course, be obtained by proceeding formally with the theory of characteristics for the quasi-linear partial differential equations which depict the flow. In the interests of simplifying both the derivation of these relations and their resultant forms, however, it seems desirable to proceed in a more intuitive manner, assuming a priori that the characteristic lines are Mach lines and streamlines (in steady flow), and utilizing the implication from two-dimensional-flow studies that perhaps the most convenient dependent variables are pressure and flow inclination angles. With this approach in mind, we employ the Euler momentum equations,

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{3}$$

the continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0 \tag{4}$$

the equation of state,

$$\rho = \rho(p,s) \tag{5}$$

and the energy equation,

$$\frac{\partial s}{\partial t} + U\frac{\partial s}{\partial x} + V\frac{\partial s}{\partial y} + W\frac{\partial s}{\partial z} = 0 \tag{6}$$

where U, V, and W are the components of velocity at time t along the x, y, and z axes, respectively, of an element of the fluid of density ρ, static pressure p, and entropy s[1] (see appendix for list of symbols). To put these expressions in a more tractable form, it is convenient to aline the x axis at time t with the direction of the resultant velocity at the origin of the coordinate system. Thus equations (1) through (4) and equation (6) simplify, respectively, in the region of the origin, to

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \tag{7}$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \tag{8}$$

$$\frac{\partial W}{\partial t} + U \frac{\partial W}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0 \tag{9}$$

$$\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial x} + \rho \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right) = 0 \tag{10}$$

and

$$\frac{\partial s}{\partial t} + U \frac{\partial s}{\partial x} = 0 \tag{11}$$

which relations are basic to the subsequent analysis.

Steady Flow

Characteristics method.- It is clear that in this case all derivatives with respect to time disappear from the above relations. Thus, assuming there are no shock waves present in the region of the origin,[2] we may write, with the aid of equations (5) and (11),

[1] For certain calculations it may be desirable to proceed from more general equations which include effects of heat and mass addition to (or subtration from) the flow as well as effects of impressed forces (e.g., gravitational or magnetic). Such a procedure may easily be developed from that presented here by following the method of Guderley (reference 6) for two-dimensional flow.

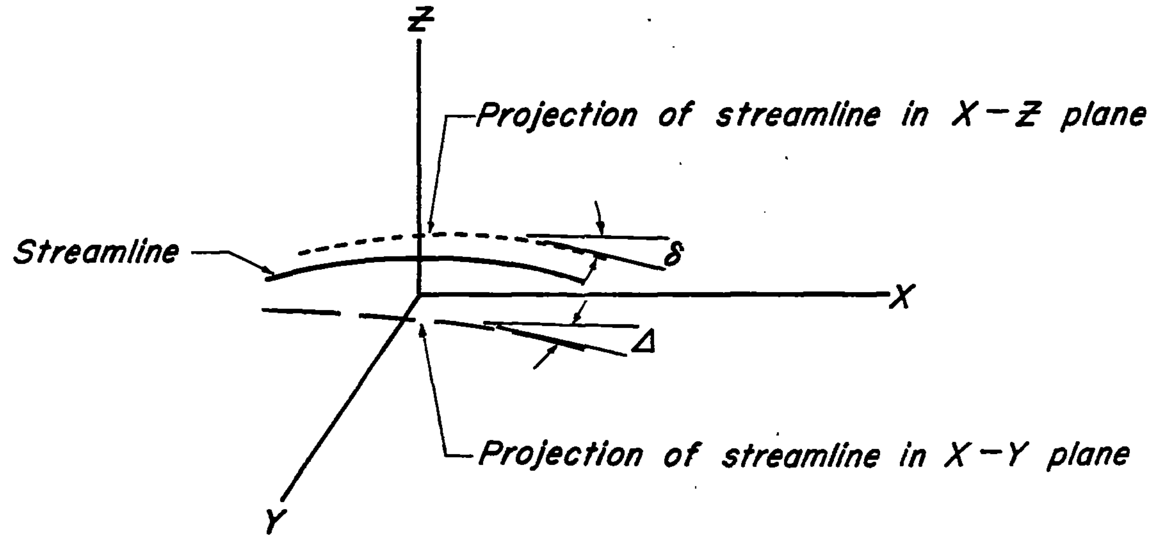
[2] If shock waves are present, the appropriate oblique shock equations are employed.

$$\frac{\partial \rho}{\partial x} = \left.\frac{\partial \rho}{\partial p}\right|_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x} \tag{12}$$

where a is the local speed of sound in the fluid. Combining equations (7), (10), and (12), there is then obtained the relation

$$\frac{\partial p}{\partial x} = \frac{-\rho U^2}{M^2-1}\left[\frac{1}{U}\left(\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right)\right] \tag{13}$$

or, defining Δ as the angle between the x axis and the tangent to the projection of a streamline in the $x - y$ plane, and, in an analogous manner, the angle δ in the $x - z$ plane (see sketch), we have



$$\frac{\partial p}{\partial x} = \frac{-\rho U^2}{M^2-1}\left(\frac{\partial \Delta}{\partial y} + \frac{\partial \delta}{\partial z}\right) \tag{14}$$

Transforming the derivatives with respect to x and z to derivatives in the characteristic or C_{1z} and C_{2z} directions in the $x - z$ plane (C_{1z} is positively inclined with respect to x, thus $\partial()/\partial x = [M/(2\sqrt{M^2-1})]$ $[\partial()/\partial C_{1z} + \partial()/\partial C_{2z}]$ and $\partial()/\partial z = (M/2)[\partial()/\partial C_{1z} - \partial()/\partial C_{2z}]$), there results from this equation

$$\frac{\partial p}{\partial C_{1z}} + \frac{\partial p}{\partial C_{2z}} = + \frac{\rho U^2}{\sqrt{M^2-1}}\left[\frac{\partial \delta}{\partial C_{2z}} - \frac{\partial \delta}{\partial C_{1z}} - \frac{2}{M}\left(\frac{\partial \Delta}{\partial y}\right)\right] \tag{15}$$

In an analogous manner, there is obtained from equation (9) the relation

$$\frac{\partial p}{\partial C_{1z}} - \frac{\partial p}{\partial C_{2z}} = \frac{-\rho U^2}{\sqrt{M^2-1}}\left(\frac{\partial \delta}{\partial C_{1z}} + \frac{\partial \delta}{\partial C_{2z}}\right) \tag{16}$$

Adding these two expressions then yields

$$\frac{\partial p}{\partial C_{1z}} = \frac{-\rho U^2}{\sqrt{M^2-1}} \left[\frac{\partial \delta}{\partial C_{1z}} + \frac{1}{M} \left(\frac{\partial \Delta}{\partial y} \right) \right] \tag{17}$$

while subtracting yields

$$\frac{\partial p}{\partial C_{2z}} = \frac{\rho U^2}{\sqrt{M^2-1}} \left[\frac{\partial \delta}{\partial C_{2z}} - \frac{1}{M} \left(\frac{\partial \Delta}{\partial y} \right) \right] \tag{18}$$

Equations (17) and (18) are compatibility equations for charateristic or Mach lines in the $x - z$ plane.[3] Indeed, if it is further required that the $x - z$ plane be the osculating plane of the streamline passing through the origin, that is, the plane containing the principal radius of curvature and tangent to this streamline (at the origin), then these equations are the essential relations for determining pressure and flow inclination throughout a flow field. This point becomes evident when it is observed that with the imposed requirement (viz., $\partial\Delta/\partial x = 0$), the additional information derived from studying flow in the $x - y$ plane is simply that deduced from equation (8), or, as would be expected,

$$\frac{\partial p}{\partial y} = 0 \tag{19}$$

In proceeding to construct a flow field, however, it is necessary to know the manner in which the osculating plane rotates and, correspondingly, how the principal curvature varies along a streamline. This information is easily obtained from equations (2) and (3) by differentiating with respect to x, thus yielding

$$\frac{\partial^2 \Delta}{\partial x^2} = - \frac{1}{\rho U^2} \frac{\partial}{\partial x} \left(\frac{\partial p}{\partial y} \right) - \frac{\partial \Delta}{\partial z} \frac{\partial \delta}{\partial x} \tag{20}$$

and

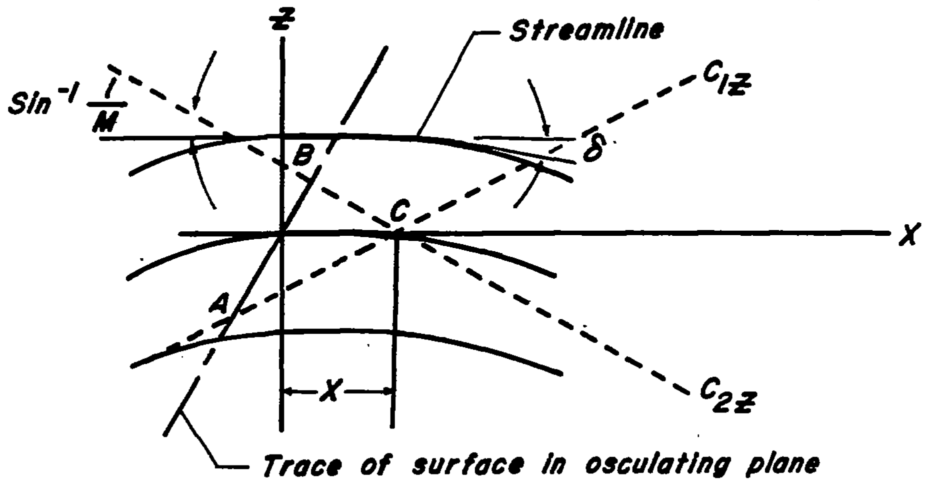
$$\frac{\partial^2 \delta}{\partial x^2} = - \frac{1}{\rho U^2} \frac{\partial}{\partial x} \left(\frac{\partial p}{\partial z} \right) - \left(\frac{\partial \delta}{\partial z} + \frac{3}{U} \frac{\partial U}{\partial x} + \frac{1}{\rho a^2} \frac{\partial p}{\partial x} \right) \frac{\partial \delta}{\partial x} \tag{21}$$

respectively.

With the aid of these and the previously derived expressions, we now consider how the characteristics method for steady three-dimensional flows

[3]It is noted that these expressions contain not only derivatives in the characteristic directions but also derivatives with respect to the independent variable y. This type of result is to be expected as pointed out by Coburn (reference 19).

can be applied. In particular, only the initial value problem will be considered here. Thus it is first assumed that flow conditions are known, or in some manner may be determined on a surface[4] in the disturbed flow field to be investigated. We then consider flow in the region of the osculating plane of one of the streamlines passing through this surface (see sketch below).



Using known conditions at points A and B, pressure and flow inclination at C are calculated employing equations (17) and (18) as difference equations. Orientation of the osculating plane and curvature of the streamline at C are then determined with the aid of equations (20) and (21) and the knowledge of x and $\partial\delta/\partial x$. Other properties, such as velocity, temperature, and Mach number are determined at point C in a manner analogous to that for two-dimensional flow (see, e.g., reference 7). Once these calculations have been carried out at a number of points, like C, close to the initial surface, then the whole procedure is repeated, progressing from these new points (or points interpolated therefrom) of known flow conditions to other points farther removed from the surface. Thus the flow field is constructed moving downstream (or upstream) from the initial surface.

Variations on these calculations are frequently required in practical applications; for example, if a shock wave is encountered in the construction, it is usually necessary to solve simultaneously the equations for the shock and the compatibility equation for the characteristic line intersecting the shock in order to determine flow conditions at a point adjacent to the shock. With an analogous procedure, points on the surface of a body may be treated. In any case, it would appear that

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[4]If no other information on the flow field is available, this surface cannot, as is known from the theory of characteristics, be a characteristic surface. However, as is frequently the case in aerodynamics, this restriction is eliminated since the initial surface intersects a shock wave or body of known shape, or both, which result provides additional boundary conditions at the terminal edges of the surface.

complication of calculation has been reduced by working with pressure and flow inclination angles as primary dependent variables, no loss in generality being sustained insofar as restrictions on the thermodynamic behavior of the gas is concerned. The gas may exhibit both thermal and caloric imperfections without invalidating the previously developed equations, the extent to which these imperfections are manifest influencing only the form of the equation of state of the fluid and the relations defining its specific heats.[5] The provision for caloric imperfections would appear (see reference 7) to be especially desirable when using the method to calculate hypersonic flows.

Application of the method to the calculation of a specific flow field hinges on the determination of flow conditions along the initial surface. This is a separate problem, the solution of which has thus far at least been special to the particular type of flow under consideration.[6] Indeed, only the airfoil has apparently been treated rigorously in this connection without the restriction that air behaves as an ideal gas (see reference 22).

Imperfect gas flow downstream of the throat of a hypersonic nozzle could, of course, be analyzed by the characteristics method of this paper. With the restriction to ideal gas flows, an additional application of importance also suggests itself, namely, to the calculation of flows about inclined bodies of revolution at angle of attack (not necessarily small). In this case flow conditions along a surface close to the vertex of the body can be calculated approximately with the aid of references 25 and 26, the accuracy of the approximation increasing with the closeness of the surface to the vertex. Flow downstream of this surface can then be determined after the manner discussed, using equations (17), (18), (20), and (21) in the reduced forms

$$\frac{\partial p}{\partial C_{1z}} = \frac{-\gamma p M^2}{\sqrt{M^2-1}}\left[\frac{\partial \delta}{\partial C_{1z}} + \frac{1}{M}\left(\frac{\partial \Delta}{\partial y}\right)\right]$$

$$\frac{\partial p}{\partial C_{2z}} = \frac{\gamma p M^2}{\sqrt{M^2-1}}\left[\frac{\partial \delta}{\partial C_{2z}} - \frac{1}{M}\left(\frac{\partial \Delta}{\partial y}\right)\right]$$

$$\frac{\partial^2 \Delta}{\partial x^2} = -\frac{1}{\gamma p M^2}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) - \frac{\partial \Delta}{\partial z}\frac{\partial \delta}{\partial x}$$

[5]Since only flows of dense air are treated here, heat capacity lag effects are considered negligible.

[6]In this connection see, for example, the work of Crocco (reference 20), Munk and Prim (reference 21), and Kraus (reference 22) on the two-dimensional airfoil problem, and Shen and Lin (reference 23) and Cabannes (reference 24) on axially symmetric flow about bodies of revolution.

and

$$\frac{\partial^2\delta}{\partial x^2} = -\frac{1}{\gamma pM^2}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) - \left[\frac{\partial\delta}{\partial z} + \frac{1}{\gamma pM^2}\frac{\partial p}{\partial x}(M^2-3)\right]\frac{\partial\delta}{\partial x}$$

respectively, where γ, the ratio of specific heat at constant pressure to specific heat at constant volume, would be considered constant (equal to approximately 1.4 for air).

Let us now turn our attention to the consideration of a more approximate, and by the same token more simplified, method of calculating the steady three-dimensional flow of a gas at high supersonic speeds.

Approximate method.- It is well at the outset of this analysis to establish, insofar as is practicable, the type of flows to be treated. In this connection, it is convenient to employ the hypersonic similarity parameter (i.e., the product of the flight Mach number and say the thickness ratio of a body) as a measuring stick. In flows characterized by values of the hypersonic similarity parameter small compared to 1, that is, flows in which the body is extremely slender and lies close to the axis of the Mach cone, there is no apparent reason to believe that the linear theory will not be as useful an approximate method of calculation as at low supersonic speeds. In flows characterized by values of the parameter up to about 1, the second-order theory first enunciated by Busemann (reference 27) for airfoils and more recently generalized to three-dimensional flows by Van Dyke (reference 9) and Moore (reference 28) should prove a useful approximation. On the other hand, for flows about more or less arbitrary shapes, there is apparently no approximate method of calculation generally applicable with engineering accuracy at values of the hypersonic similarity parameter appreciably greater than 1.

In the limiting case of indefinitely high free-stream Mach number (and hence similarity parameter) and a ratio of specific heats equal to 1, we have the Newtonian impact theory (reference 29) and its refined counterpart, accounting for centrifugal forces in the disturbed flow, developed first by Busemann (reference 30) and more recently treated by Ivey, Klunker, and Bowen (reference 31). The impact theory has been employed with some success by Grimminger, Williams, and Young (reference 32) and others to predict surface pressures on bodies of revolution at values of the similarity parameter appreciably greater than 1, although it should be remarked in passing that this success is in part, at least, fortuitous, as perhaps is best evidenced by the fact that the more exact theory (within the framework of the underlying assumption of $M \to \infty$, $\gamma \to 1$) of Busemann is considerably less accurate under corresponding circumstances. As shown in reference 7, neither the Newtonian impact nor the Busemann theory apply with good accuracy to airfoils except at values of the similarity parameter quite large compared to 1, corresponding, for example, in the case of thin airfoils to flight speeds considerably in excess of the escape speed at sea level. Perhaps the foremost shortcoming of these theories is, however, that, irrespective of the shape to which they are applied, they provide no information on the structure[7] of the disturbed flow field which is, of course, of finite extent adjacent to the surface at flight Mach numbers presently of interest (say Mach numbers less than the escape Mach number at sea level). Such information is, for example, important to the determination of the flow about control surfaces and the like which may be located in this field.

In view of the preceding discussion, it seems clear that in the high-supersonic-speed flight regime, the need for an approximate method of analysis lies in the realm of flows characterized by values of the hypersonic similarity parameter greater than 1. An attempt will therefore be made to obtain a method meeting part of this need, attention being focused primarily on flows characterized by large values of the similarity parameter. To this end, it is convenient first to employ equation (14) rewritten in the form

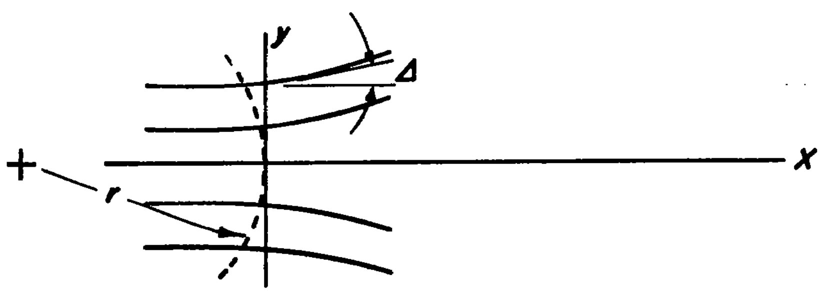
$$\frac{\partial p}{\partial x} = \frac{\rho U^2}{\sqrt{M^2-1}}\left[\frac{\partial \delta}{\partial x}\left(\frac{1 - D_z}{1 + D_z}\right) - \frac{1}{\sqrt{M^2-1}}\left(\frac{\partial \Delta}{\partial y}\right)\right] \tag{22}$$

where

$$D_z = \frac{\partial \delta / \partial C_{1z}}{\partial \delta / \partial C_{2z}} \tag{23}$$

Now consider for the moment a surface streamline alined in the x direction, and impose the requirement that the $x - z$ plane be tangent to this streamline and normal to the surface at the point of tangency (the origin). The $x - y$ plane is then, of course, tangent to the surface at this point. Observing the last term in the brackets on the right-hand side of equation (22), it is noted (see sketch) that

$$\frac{\partial \Delta}{\partial y} = \frac{1}{r}$$



[7] This consequence is traceable primarily to the assumption of $\gamma = 1$ which leads to the well-known result that the disturbed flow field is confined to an infinitesimal region adjacent to the surface of a body.

where r is the radius of curvature of the line normal to the projections of streamlines in the $x - y$ plane, and passing through the origin. At the high Mach numbers under consideration, the disturbed flow field is confined to a region of small extent normal to the surface of a body; hence it may be expected that r will be primarily a function of body shape and attitude.[8] This being the case, it follows then that the term $(1/\sqrt{M^2-1})(1/r)$ will decrease in absolute magnitude with increasing Mach number of the flow about the body. Consider now the term $(\partial\delta/\partial x)(1-D_z)/(1+D_z)$. We note that $\partial\delta/\partial x = 1/R$ where R is the radius of curvature of the projection of a streamline in the $x - z$ plane and, by reasoning analogous to that used in considering r, is not expected to vary significantly with Mach number in the disturbed flow field. Let us assume for the moment that the quantity $(1-D_z)/(1+D_z)$ is also relatively independent of Mach number. With this assumption, it is clear that equation (22) approaches the equation for two-dimensional flow as the free-stream Mach number, and hence the hypersonic similarity parameter of the flow becomes large compared to 1. The compatibility equations (equations (17) and (18)) are affected in a similar manner; thus it is apparent that the flow when viewed in the $x - z$ plane approaches the two-dimensional type. In this case, however, as shown in reference 7, so long as the Mach number and ratio of specific heats of the disturbed fluid are not too close to 1, D_z is small compared to 1, and hence the flow approaches the generalized Prandtl-Meyer type (i.e., flow in which pressure and inclination angle are approximately constant along curved first-family Mach lines). Our flow equation may then be written

$$\frac{\partial p}{\partial x} \cong \frac{\rho U^2}{\sqrt{M^2-1}}\left(\frac{\partial\delta}{\partial x}\right) \tag{24}$$

where it is required explicitly that

$$\left|\frac{\partial\delta}{\partial x}\right| \gg \frac{1}{\sqrt{M^2-1}}\left|\frac{\partial\Delta}{\partial y}\right| \tag{25}$$

or, in effect, that disturbances associated with the divergence of streamlines in tangent planes must be of secondary importance compared to those associated with the curvature of streamlines in planes normal to the surface.

From these considerations it appears that the conclusion of reference 10 that inviscid flow along streamlines downstream of the nose of

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[8]It is interesting to note that in ideal gas flows, r becomes just a function of these variables as the value of the hypersonic similarity parameter becomes large compared to 1 (see work of Oswatitsch, reference 33, noting that his results can readily be extended to three-dimensional ideal gas flows using the characteristics method of this paper).

noninclined bodies of revolution traveling at high supersonic speeds is often of the Prandtl-Meyer type (in regions free of shock waves) may apply also to other steady three-dimensional flows. It is true, too, that in the latter case, just as in the former case, this conclusion is consistent with the predictions of the hypersonic similarity law for steady flow about slender shapes.

One question remains to be considered, namely, where do the streamlines go in the disturbed flow? To clarify this matter, it is convenient to study further the implications of equation (25). For this purpose we combine equation (25) with the transformation equation

$$\frac{\partial\Delta}{\partial y} = \frac{M}{2}\left(\frac{\partial\Delta}{\partial C_{1y}} - \frac{\partial\Delta}{\partial C_{2y}}\right)$$

to obtain the relation

$$\left|\frac{\partial\delta}{\partial x}\right| \gg \frac{M}{2\sqrt{M^2-1}}\left|\frac{\partial\Delta}{\partial C_{1y}} - \frac{\partial\Delta}{\partial C_{2y}}\right| \qquad (26)$$

From this relation we deduce either that to the order of a number (curvature) small compared to $\frac{2\sqrt{M^2-1}}{M}\left|\frac{\partial\delta}{\partial x}\right|$

$$\frac{\partial\Delta}{\partial C_{1y}} \cong \frac{\partial\Delta}{\partial C_{2y}} \qquad (27)$$

or that

and

$$\left.\begin{aligned}\left|\frac{\partial\delta}{\partial x}\right| &\gg \frac{M}{2\sqrt{M^2-1}}\left|\frac{\partial\Delta}{\partial C_{1y}}\right| \\ \left|\frac{\partial\delta}{\partial x}\right| &\gg \frac{M}{2\sqrt{M^2-1}}\left|\frac{\partial\Delta}{\partial C_{2y}}\right|\end{aligned}\right\} \qquad (28)$$

Equation (27) implies vortical flow, however, which type of flow cannot be treated by the present analysis since equation (25) is violated.[9] Equation (28) is then the requirement consistent with the basic assumptions

[9]This conclusion is particularly evident in the case of pure vortical flow, or say vortical flow with a superimposed uniform stream directed along the axis of the vortex, in which cases $\partial p/\partial x = 0$, and hence equation (24) certainly does not follow from equation (22).

of this analysis. Comparing the relations of equation (28) with the transformation equation

$$\frac{\partial\Delta}{\partial x} = \frac{M}{2\sqrt{M^2-1}}\left(\frac{\partial\Delta}{\partial C_{1y}} + \frac{\partial\Delta}{\partial C_{2y}}\right)$$

leads one to the conclusion, however, that

$$\left|\frac{\partial\delta}{\partial x}\right| \gg \left|\frac{\partial\Delta}{\partial x}\right| \tag{29}$$

or, in effect, that surface streamlines can, to the accuracy of this analysis, be considered geodesic lines of the surface. With this infortion we are enabled to construct the flow field about a body, having once determined, for example, the flow in the region of the leading edge (or edges) thereof. This result follows since a geodesic line, and hence a streamline, on the surface is fixed, provided its direction at any point is given (see, e.g., reference 34).[10] With this knowledge of the location of surface streamlines, flow in the planes tangent thereto and normal to the surface may be calculated approximately in the relatively thin region between the surface and bounding shock waves, using the generalized shock-expansion method after the manner described in reference 7.

A partial check on these observations is afforded by studying the flow about a swept airfoil. In this case flow at the surface may be calculated with good accuracy, using the shock-expansion method in combination with simple-sweep theory. For thin airfoils (on the surface of which the appropriate geodesics have essentially the direction of the free stream) the extended shock-expansion method of this paper reduces to the slender-airfoil method of reference 7. Thus, in this case, it is evident from the results of reference 7 that the extended method will predict surface pressure coefficients in error by less than 10 percent, providing the component of free-stream Mach number normal to the leading edge is greater than about 3. It is of interest also to consider a thick airfoil to ascertain the accuracy with which this method applies to flow with appreciable curvature. To this end, surface pressure coefficients and streamlines have been calculated for a 20-percent-thick biconvex airfoil (at zero incidence) swept 60° and operating at Mach numbers of 10 and infinity ($\gamma = 1.4$). The results of these calculations are presented in figure 1, and it is observed that the pressure distributions determined with the shock-expansion method for swept airfoils and the extended shock-expansion method are in reasonably good agreement at both Mach numbers.

[10] If a sudden change of surface slope causes an oblique shock wave or a concentrated Prandtl-Meyer type expansion fan, the streamlines in the downstream direction are defined on the basis of their flow direction immediately following the discontinuity in slope.

The streamlines are also in reasonably good agreement over the forward portion of the airfoil, although, as would be expected, somewhat poorer results are obtained over the afterportion. It is not surprising, in view of the underlying assumptions of the extended shock-expansion method, that it is generally more accurate at the highest Mach number.

In the preceding discussion circumstances were deduced under which steady flow at high supersonic speeds about three-dimensional shapes could be constructed approximately, using the basic tools of two-dimensional supersonic flow analysis, namely, the oblique shock equations and Prandtl-Meyer equations. Several possible exceptions to these circumstances immediately come to mind. These include conical-type flow and flow in the region of the tip of a wing, or at the discontinuous juncture of a wing and body, to mention a few. In such flows equation (25) may not be satisfied, in which case two-dimensional flow in planes normal to a surface cannot be expected.[11] It might be reasoned, therefore, that these flows cannot, in general, be treated by the proposed method. This observation may be correct; however, in the one case investigated thus far in this connection, namely, flow in the region of the nose of non-lifting bodies of revolution (see reference 10), it was found that although equation (25) is not satisfied, flow along streamlines is nevertheless of approximately the Prandtl-Meyer type. Thus we are led to expect that perhaps a less restrictive requirement than the satisfying of equation (25) may be imposed to insure that flow along streamlines is of this type. Such a requirement is in fact easily obtained by reconsidering equation (22) in the form

$$\frac{\partial p}{\partial x} = \frac{\rho U^2}{\sqrt{M^2-1}}\left[\frac{\partial \delta}{\partial x} - \frac{M}{\sqrt{M^2-1}}\left(\frac{\partial \delta}{\partial C_{1z}} + \frac{1}{M}\frac{\partial \Delta}{\partial y}\right)\right] \tag{30}$$

thus yielding

$$\left|\frac{\partial \delta}{\partial x}\right| \gg \frac{M}{\sqrt{M^2-1}}\left|\frac{\partial \delta}{\partial C_{1z}} + \frac{1}{M}\frac{\partial \Delta}{\partial y}\right| \tag{31}$$

It is evident that equation (31) embraces equation (25) as a special case and that Prandtl-Meyer flow obtains along streamlines if

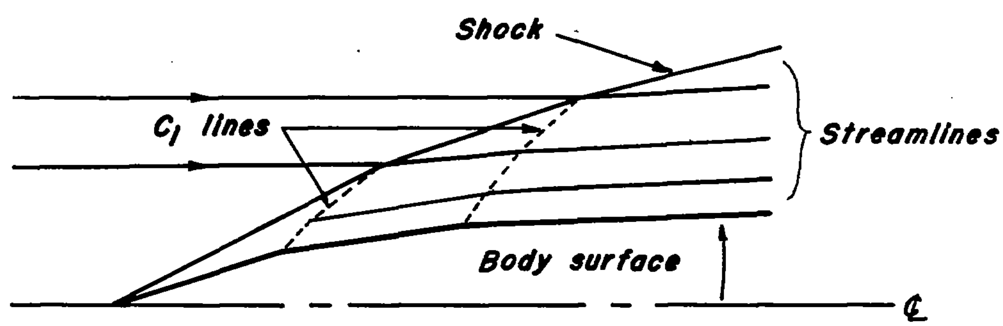
$$\frac{\partial \delta}{\partial C_{1z}} \cong -\frac{1}{M}\frac{\partial \Delta}{\partial y} \tag{32}$$

---

[11] One may note that in some cases of this nature, the flow in osculating planes of the streamlines may be of the two-dimensional or even the simpler Prandtl-Meyer type, although these planes may not be normal to the surface.

to the order of a number small compared to $\frac{\sqrt{M^2-1}}{M}\left|\frac{\partial\delta}{\partial x}\right|$. This result implies that although flow inclination angles are not necessarily constant along C_{1z} lines, pressure is approximately constant (see equation (17)).

It is clear that the increased generality of the above result has been obtained at some expense in our knowledge of the streamline flow pattern. For example, it is not now indicated that (within the framework of this analysis) surface streamlines may generally be taken as geodesics - additional knowledge of the flow must be had in order to determine these streamlines. If they are known, however, the calculation of the whole flow field is materially facilitated by the above considerations. To illustrate, consider a nonlifting body of revolution (see sketch)



for which we assume that flow at the vertex is known either from say reference 8 or reference 10. The meridian curve of the body is broken up into short segments as shown, and flow is constructed along the first-family Mach lines emanating from the intersection of these segments. The requirement to be satisfied is that the pressure change across these lines be constant along their length.[12] The construction proceeds then in a manner analogous to that for the two-dimensional airfoil discussed in reference 7.

Thus far only steady flows have been considered. The problem naturally arises of extending these considerations to nonsteady flows. Some aspects of this matter will now be discussed.

Nonsteady Flow

The methods of analysis in this case are entirely analogous to those employed in the study of steady flow, the singular contrasting feature

[12]In this manner small changes in pressure along C_1 lines can be accounted for approximately in the predominantly conical flow near the vertex of the body.

being that derivatives with respect to time in equations (1) through (11) cannot now be neglected. With this point in mind, only pertinent results are discussed below.

Characteristics method.- The compatibility equations relating fluid properties along Mach lines may be written as follows:

$$\frac{\partial p}{\partial C_{1z}} = \frac{-\rho U^2}{\sqrt{M^2-1}} \left\{ \frac{\partial \delta}{\partial C_{1z}} + \frac{1}{M}\left(\frac{\partial \Delta}{\partial y}\right) + \frac{1}{U}\left[\frac{\sqrt{M^2-1}}{M}\left(\frac{\partial \delta}{\partial t}\right) + \frac{M}{\rho U^2}\left(\frac{\partial p}{\partial t}\right) - \frac{1}{MU}\left(\frac{\partial U}{\partial t}\right)\right]\right\} \tag{33}$$

and

$$\frac{\partial p}{\partial C_{2z}} = \frac{+\rho U^2}{\sqrt{M^2-1}} \left\{ \frac{\partial \delta}{\partial C_{2z}} - \frac{1}{M}\left(\frac{\partial \Delta}{\partial y}\right) + \frac{1}{U}\left[\frac{\sqrt{M^2-1}}{M}\left(\frac{\partial \delta}{\partial t}\right) - \frac{M}{\rho U^2}\left(\frac{\partial p}{\partial t}\right) + \frac{1}{MU}\left(\frac{\partial U}{\partial t}\right)\right]\right\} \tag{34}$$

The definition of the $x - z$ plane as the osculating plane of a pathline (streamline in steady flow) remains as before, hence equation (19) still applies in the $x - y$ plane in the region of the origin. The rotation of the osculating plane and variation of the principal curvature of a pathline with motion along it are now, however, obtained with the aid of the relations

$$\frac{d^2\Delta}{dx^2} = -\frac{1}{\rho U^2}\frac{d}{dx}\left(\frac{\partial p}{\partial y}\right) - \frac{\partial \Delta}{\partial z}\left(\frac{d\delta}{dx}\right) \tag{35}$$

and

$$\frac{d^2\delta}{dx^2} = -\frac{1}{\rho U^2}\frac{d}{dx}\left(\frac{\partial p}{\partial z}\right) - \left(\frac{\partial \delta}{\partial z} + \frac{3}{U}\frac{dU}{dx} + \frac{1}{\rho a^2}\frac{dp}{dx}\right)\frac{d\delta}{dx} \tag{36}$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{U}\left(\frac{\partial}{\partial t}\right) \tag{37}$$

These equations in combination with the energy and state equations are employed in the same manner as in the case of steady flow to construct a flow field, proceeding from an initial value surface. It is clear, however, that because of the unsteady nature of the flow, this surface is not necessarily fixed in space, nor are fluid properties necessarily constant on it. Thus, in order to construct the flow, it will, in general, be necessary to start new osculating planes from this surface at short intervals of time, each plane being attached to a particular element of fluid as it moves through the field. By way of comparison, then, we recall that in going from steady two-dimensional to three-dimensional flow with the characteristics method, it was necessary to construct the flow in a family of surfaces (located adjacently in say the spanwise direction) rather than just a single surface. Analogously, in going from three-dimensional steady to three-dimensional nonsteady flow, it is necessary to construct the flow in a family of spaces located adjacently in the "time" direction, rather than in just one space. Quite obviously such a series of calculations poses so formidable and time consuming a problem as to be questionably feasible at present; hence they will be considered in no greater detail here. Rather, let us turn our attention to the approximate method of calculating nonsteady flows.

Approximate method.- As in the case of the corresponding steady-flow analysis, it is convenient to consider the expression for pressure gradient along a pathline. Thus we have

$$\frac{dp}{dx} = \frac{\rho U^2}{\sqrt{M^2-1}}\left[\left[\frac{d\delta}{dx} - \frac{1}{U}\left(\frac{\partial\delta}{\partial t}\right)\right]\left(\frac{1-D_z}{1+D_z}\right) - \frac{1}{\sqrt{M^2-1}}\left\{\frac{\partial\Delta}{\partial y} + \frac{1}{U}\left[\frac{1}{\rho U^2}\left(\frac{\partial p}{\partial t}\right) - \frac{\partial U}{\partial t}\left(\frac{1}{U}\right)\right]\right\}\right] \tag{38}$$

where now the $x - z$ planes are taken normal to the surface swept out by elements of fluid moving along the body. Upon inspection of this relation and the compatibility equations, it becomes clear that the critical requirement for two-dimensional flow of the generalized Prandtl-Meyer type in these planes is, in addition to the one previously derived from steady flow considerations, that

$$\frac{1}{U}\left|\frac{\partial\delta}{\partial t}\right| \ll \left|\frac{d\delta}{dx}\right|$$

or

$$\frac{1}{M^o}\left|\frac{\partial\delta}{\partial x_o}\right| \ll \left|\frac{d\delta}{dx}\right| \tag{39}$$

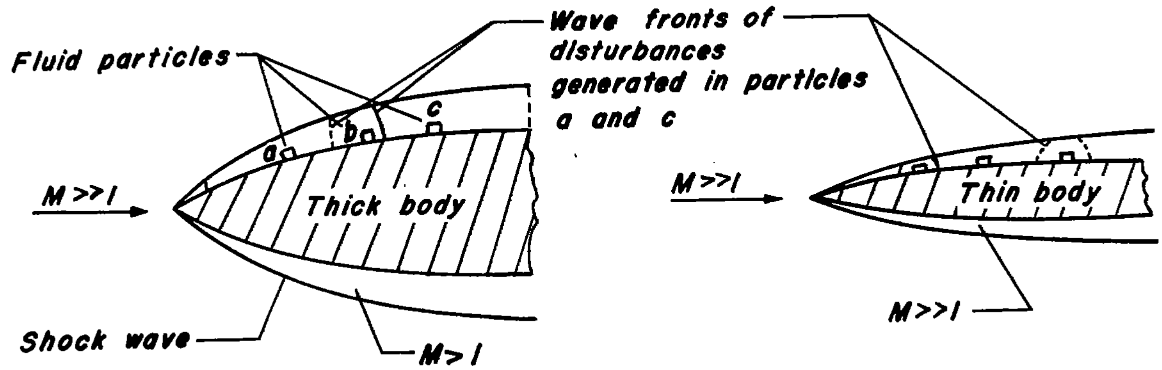
where

$$M^o = \frac{U}{a_o} \qquad x_o = a_o t$$

a_o being the velocity of sound in the undisturbed stream. Now so long as the Mach number of the undisturbed stream and the local Mach numbers are large compared to 1, M^o is large compared to 1 since the speed of the undisturbed stream and the speed of the local flow cannot differ greatly. Thus, with this restriction, the requirement expressed by equation (39) is simply that the induced curvature of the flow $|\partial\delta/\partial x_o|$, associated with the nonsteady motions of the fluid, cannot exceed in order of magnitude the total curvature $|d\delta/dx|$ of the flow. Provided this is the case, and equation (25) is satisfied, equation (38) reduces, of course, to

$$\frac{dp}{dx} \cong \frac{\rho U^2}{\sqrt{M^2-1}} \left(\frac{d\delta}{dx}\right) \tag{40}$$

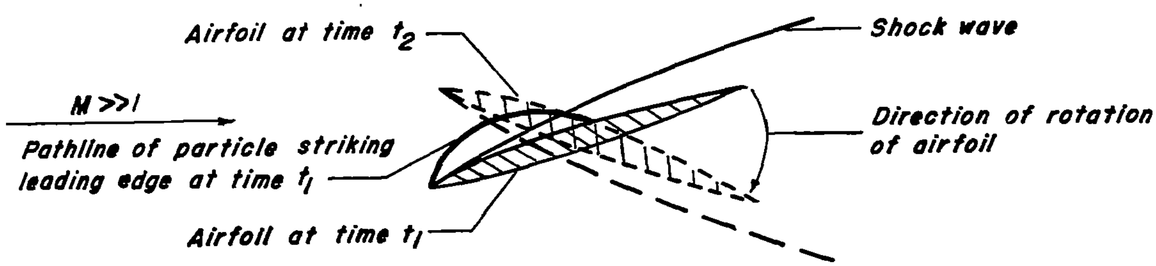
One observes that this result is not applicable to as wide a class of shapes as that for steady flow, since the local Mach number of the disturbed flow is now required to be everywhere large compared to 1.[13] This additional requirement manifests itself, since otherwise, nonsteady disturbances created an appreciable distance upstream and/or downstream of a particle could significantly influence its behavior in the disturbed flow field (see sketch, noting that in case of thick body, particle b is influenced by disturbances originating in particles a and c).



Thus equation (40) applies only in disturbed flow fields about thin or slender shapes (i.e., shapes producing flow deflection angles small compared to 1). In such cases, pathlines in the surfaces swept out by elements of fluid adjacent to the shapes are approximated by geodesics or, even simpler, lines of curvature of these surfaces. It is not to be implied, of course, that pathlines must always be such curves in order for fluid properties to behave as in Prandtl-Meyer flow. In fact, just as in the case of steady flow, if equation (31) rather than equation (25) is satisfied, pathlines are not necessarily geodesics (or lines of curvature) although equation (39) and hence equation (40) hold along these lines.

[13]The net simplification of requiring only that the hypersonic similarity parameter of the flow be large compared to 1, is, in general, that flow in osculating planes may be treated as nonsteady, two-dimensional.

One notes that within the framework of this approximate analysis, the calculation of nonsteady flows at least at the surface of slender bodies traveling at high supersonic speeds should not prove unduly difficult. To illustrate, consider an oscillating airfoil as shown in the sketch:



The pressure at any point along the pathline shown is readily deduced by simply integrating equation (40) along this line from the leading edge of the airfoil to the point in question. The whole flow field as a function of time may be calculated by employing the generalized shock-expansion method for steady flows (see reference 7) in a series of planes located small distances apart in time. This example serves to emphasize that the time history of fluid elements must be known, at least to the extent of fixing their initial flow direction and entropy. It is also evident that again, as in the case of steady flow, the general results of the analysis are consistent with the predictions of the hypersonic similarity law for nonsteady flows about slender related shapes (reference 5).

CONCLUDING REMARKS

A method of characteristics for solving steady three-dimensional supersonic flow problems has been considered. It was found that compatibility equations relating fluid properties along characteristic lines could be obtained in a simple form by employing pressure and flow inclination angles as dependent variables. No significant restrictions were imposed on either the equation of state obeyed by the fluid, or the relations defining its specific heats. These features of generality were retained for the specific purpose of enabling more accurate application of the method to the calculation of flow fields about missiles traveling at high supersonic airspeeds. Such application requires, of course, a predetermined knowledge of fluid properties along some surface in the disturbed flow. Extension of the method to treat nonsteady flows was considered briefly.

It was also undertaken to obtain an approximate method for calculating flows about bodies traveling at high supersonic speeds. It was found that when the flight Mach number is sufficiently large compared to 1, flow in the osculating planes of streamlines in regions free of shock waves may frequently be of the generalized Prandtl-Meyer type - surface streamlines in this event may be taken as geodesics. In the case of slender shapes, these results apply to nonsteady as well as steady flows, provided the induced curvature of streamlines does not exceed the total curvature in order of magnitude. It is concluded from these and other considerations that two-dimensional-flow equations may be applicable to a relatively wide class of flows, and hence configurations, at high supersonic speeds.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Aug. 15, 1952

APPENDIX

SYMBOLS

a	local speed of sound
c	chord of airfoil (measured normal to leading edge)
C_{1z}, C_{2z}	characteristic coordinates in $x - z$ plane (C_{1z} is positively inclined with respect to x)
C_p	pressure coefficient $\left(\frac{p-p_o}{\rho_o U_o^2/2}\right)$
M	Mach number (ratio of local velocity to local speed of sound)
M^o	Mach number (ratio of local velocity to speed of sound in the undisturbed stream)
p	static pressure
s	entropy
t	time
U,V,W	components of fluid velocity along the x, y, and z axes, respectively
x,y,z	rectangular coordinates
γ	ratio of specific heat at constant pressure to specific heat at constant volume
δ	angle between x axis and tangent to projection of streamline (or pathline) in $x - z$ plane
Δ	angle between x axis and tangent to projection of streamline (or pathline) in $x - y$ plane
ρ	density

Subscript

o	free-stream conditions

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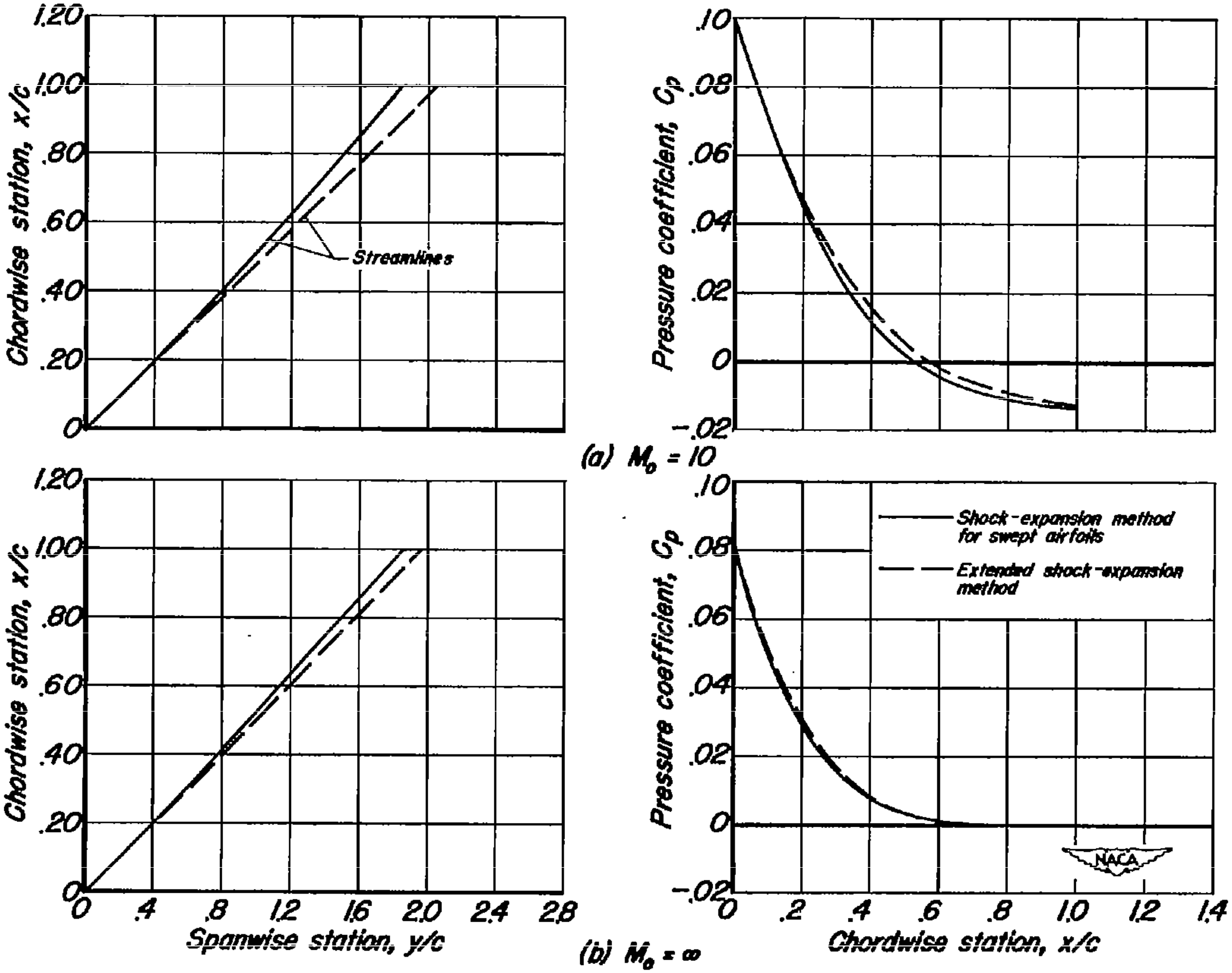

Figure 1.- Comparison of surface streamlines and pressure distribututions calculated with the extended shock-expansion method and the shock-expansion method for swept airfoils (biconvex airfoil section, thickness ratio = 0.2, sweep angle = 60°).